IB DIPLOMA PROGRAMME
PROGRAMME DU DIPLÔME DU BI
PROGRAMA DEL DIPLOMA DEL BI

MATHEMATICS
HIGHER LEVEL
PAPER 2

Friday 7 May 2004 (morning)

3 hours

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all five questions from Section A and one question from Section B.
- Unless otherwise stated in the question, all numerical answers must be given exactly or to three significant figures.
- Write the make and model of your calculator in the appropriate box on your cover sheet *e.g.* Casio *fx-9750G*, Sharp EL-9600, Texas Instruments TI-85.

Please start each question on a new page. You are advised to show all working, where possible. Where an answer is wrong, some marks may be given for correct method, provided this is shown by written working. Solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer.

SECTION A

Answer all ***five*** *questions from this section.*

1. *[Maximum mark: 13]*

The points $\mathrm{A}(1, 2)$ and $\mathrm{B}(4, 5)$ are mapped to $\mathrm{A}'(2, 3)$ and $\mathrm{B}'(5, 6)$ respectively by a linear transformation $\boldsymbol{M}$.

(a) (i) Find the matrix $\boldsymbol{M}$ which represents this transformation.

(ii) Find the image of A' under $\boldsymbol{M}$. *[7 marks]*

The point $\mathrm{C}(1, 3)$ is mapped to $\mathrm{C}'(2, 2)$ by a translation $\boldsymbol{T}$.

(b) Find the vector which represents $\boldsymbol{T}$. *[2 marks]*

(c) Find the image of $\mathrm{D}(5, 7)$ under the following transformations.

(i) $\boldsymbol{T}$ followed by $\boldsymbol{M}$;

(ii) $\boldsymbol{M}$ followed by $\boldsymbol{T}$. *[4 marks]*

2. *[Maximum mark: 14]*

(i) Jack and Jill play a game, by throwing a die in turn. If the die shows a 1, 2, 3 or 4, the player who threw the die wins the game. If the die shows a 5 or 6, the other player has the next throw. Jack plays first and the game continues until there is a winner.

(a) Write down the probability that Jack wins on his first throw. *[1 mark]*

(b) Calculate the probability that Jill wins on her first throw. *[2 marks]*

(c) Calculate the probability that Jack wins the game. *[3 marks]*

(ii) Let $f(x)$ be the probability density function for a random variable X, where

$$f(x) = \begin{cases} kx^2, & \text{for } 0 \le x \le 2 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Show that $k = \frac{3}{8}$. *[2 marks]*

(b) Calculate

(i) $\mathrm{E}(X)$;

(ii) the median of X. *[6 marks]*

3. *[Maximum mark: 14]*

(a) Consider two unit vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in three-dimensional space. Prove that the vector $\boldsymbol{u} + \boldsymbol{v}$ bisects the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$. *[4 marks]*

Consider the points $\mathrm{A}(2, 5, 4)$, $\mathrm{B}(1, 3, 2)$ and $\mathrm{C}(5, 5, 6)$. The line l passes through B and bisects angle ABC.

(b) Find an equation for l. *[4 marks]*

(c) The line l meets (AC) at the point D. Find the coordinates of D. *[6 marks]*

4. *[Maximum mark: 10]*

Let $f(x) = x\cos x$, for $0 \le x \le \pi$. The curve of $f(x)$ has a local maximum at $x = a$ and a point of inflexion at $x = b$.

(a) Sketch the graph of $f(x)$ indicating the approximate positions of a and b. *[2 marks]*

(b) Find the value of

(i) a;

(ii) b. *[3 marks]*

(c) Use integration by parts to find an expression for $\int x\cos x \, \mathrm{d}x$. *[3 marks]*

(d) Hence find the **exact** value of the area enclosed by the curve and the x-axis, for $0 \le x \le \frac{\pi}{2}$. *[2 marks]*

5. *[Maximum mark: 19]*

(a) Show that $\cos(A + B) + \cos(A - B) = 2\cos A\cos B$. *[2 marks]*

(b) Let $T_n(x) = \cos(n \arccos x)$ where x is a real number, $x \in [-1, 1]$ and n is a positive integer.

(i) Find $T_1(x)$.

(ii) Show that $T_2(x) = 2x^2 - 1$. *[5 marks]*

(c) (i) Use the result in part (a) to show that $T_{n+1}(x) + T_{n-1}(x) = 2xT_n(x)$.

(ii) Hence or otherwise, prove by induction that $T_n(x)$ is a polynomial of degree n. *[12 marks]*

SECTION B

*Answer **one** question from this section.*

Statistics

6. *[Maximum mark: 30]*

(i) Charles knows from past experience that the number of letters per day delivered to his house by the postman follows a Poisson distribution with mean 3.

(a) On a randomly chosen day, find the probability that two letters are delivered. *[2 marks]*

(b) On another day, Charles sees the postman approaching his house so he knows that he is about to receive a delivery. Calculate the probability that he receives two letters on this day. *[3 marks]*

(ii) The following is a random sample of 16 observations from a normal distribution with mean μ.

16.9	15.0	15.8	18.7	20.9	19.9	18.6	22.1
16.5	18.8	16.4	17.7	16.1	17.2	20.6	19.8

Calculate a 95 % confidence interval for μ. *[6 marks]*

(iii) Seven coins are thrown simultaneously 320 times. The results are shown in the table below.

Number of heads obtained	Frequency
1	12
2	43
3	79
4	86
5	65
6	29
7	6

The null hypothesis H_0 is "six of the coins are fair and the other coin has two heads".

(a) State, in words, the alternative hypothesis H_1. *[2 marks]*

(b) Determine, at the 5 % significance level, whether analysis of the above data results in the acceptance or rejection of H_0. *[7 marks]*

(This question continues on the following page)

(Question 6 continued)

(iv) The heights, in cm, of men and women in a random sample were measured with the following results.

Heights of men (cm)	183	171	167	169	175	181	179	171	165
Heights of women (cm)	152	169	156	163	158	161	160	162	

It is believed that the mean height of the men exceeds the mean height of the women by more than 10 cm. Use a one-tailed test at the 10 % level of significance to investigate whether this is true. You may assume that heights of men and women are normally distributed with the same variance. *[10 marks]*

Sets, Relations and Groups

7. *[Maximum mark: 30]*

(i) The relation R is defined on the points $\mathrm{P}(x, y)$ in the plane by

$$(x_1, y_1) R (x_2, y_2) \text{ if and only if } x_1 + y_2 = x_2 + y_1 .$$

(a) Show that R is an equivalence relation. *[4 marks]*

(b) Give a geometric description of the equivalence classes. *[2 marks]*

(ii) The binary operation $*$ is defined for $x, y \in \mathbb{R}$ by

$$x * y = xy - x - y + 2 .$$

(a) Find the identity element of $*$. *[2 marks]*

(b) Find the inverse of 3 under $*$. *[2 marks]*

(c) (i) Show that

$$(x * y) * z = xyz - yz - zx - xy + x + y + z .$$

(ii) Determine whether or not $*$ is associative. *[6 marks]*

(This question continues on the following page)

(Question 7 continued)

(iii) Consider the set $S = \{1, 3, 5, 7, 9, 11, 13, 15\}$ under the operation $\otimes$, multiplication modulo 16.

(a) Calculate

(i) $3 \otimes 5$;

(ii) $3 \otimes 7$;

(iii) $9 \otimes 11$. *[3 marks]*

(b) (i) Copy and complete the operation table for S under $\otimes$.

$\otimes$	**1**	**3**	**5**	**7**	**9**	**11**	**13**	**15**
1	1	3	5	7	9	11	13	15
3	3	9				1	7	13
5	5			3	13	7	1	11
7	7		3		15	13	11	9
9	9		13	15	1		5	7
11	11	1	7	13			15	5
13	13	7	1	11	5	15	9	3
15	15	13	11	9	7	5	3	1

(ii) Assuming that $\otimes$ is associative, show that $(S, \otimes)$ is a group. *[5 marks]*

(c) Find all elements of order

(i) 2;

(ii) 4. *[4 marks]*

(d) Find a cyclic sub-group of order 4. *[2 marks]*

Discrete Mathematics

8. *[Maximum mark: 30]*

(i) Find the general solution of the difference equation,

$$x_{n+2} = 3x_{n+1} + 28x_n \,,\; x_0 = 7 \,,\; x_1 = -6 \,, \text{ for } n = 0, 1, 2, \ldots .$$ *[5 marks]*

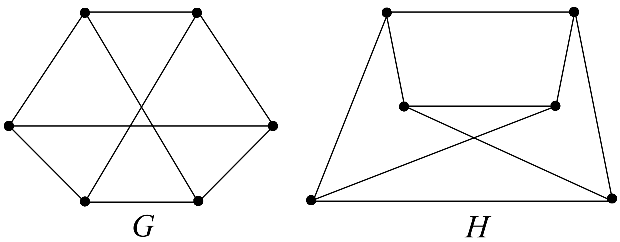
(ii) (a) Define the following terms.

(i) A bipartite graph.

(ii) An isomorphism between two graphs, *M* and *N*. *[4 marks]*

(b) Prove that an isomorphism between two graphs maps a cycle of one graph into a cycle of the other graph. *[3 marks]*

(c) The graphs *G*, *H* and *J* are drawn below.



(i) Giving a reason, determine whether or not *G* is a bipartite graph.

(ii) Giving a reason, determine whether or not there exists an isomorphism between graphs *G* and *H*.

(iii) Using the result in part (b), or otherwise, determine whether or not graph *H* is isomorphic to graph *J*. *[7 marks]*

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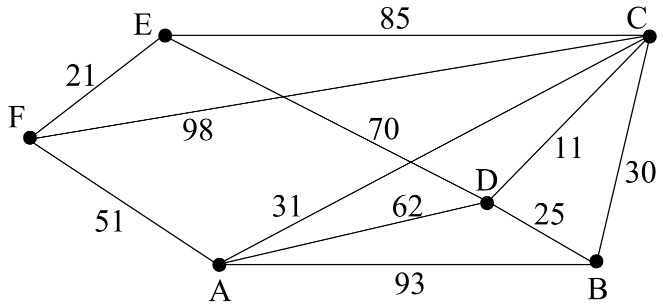
(Question 8 continued)

(iii) The following diagram shows a weighted graph.



(a) Use Kruskal's algorithm to find a minimal spanning tree for the graph. *[4 marks]*

(b) Draw the minimal spanning tree and find its weight. *[2 marks]*

(iv) (a) State the well-ordering principle. *[2 marks]*

(b) Use the well-ordering principle to prove that, given any two positive integers a and $b, (a < b)$, there exists a positive integer n such that $na > b$. *[3 marks]*

Analysis and Approximation

9. *[Maximum mark: 30]*

(i) Determine whether the following series is convergent or divergent.

$$\sum_{k=1}^{\infty} \cos\left[\frac{(k-1)\pi}{2k}\right]$$ *[5 marks]*

(ii) Let $f : x \mapsto \sin x,\ x \in \left[0, \frac{\pi}{2}\right]$.

(a) Let A be the area enclosed by the graph of f, the x-axis and the line $x = \frac{\pi}{2}$. Find the value of A. *[2 marks]*

(b) Using Simpson's Rule, find an approximation to A with an error less than 10^{-4}. *[6 marks]*

(c) Check that the error is less than 10^{-4}. *[1 mark]*

(iii) (a) Use the mean value theorem to prove that, for all $x \in \left[0, \frac{\pi}{2}\right]$,

$$\sin x \le x .$$ *[2 marks]*

(b) Hence, or otherwise, prove that for all $x \in \left[0, \frac{\pi}{2}\right]$,

$$\sin x \ge x - \frac{x^3}{6} .$$ *[4 marks]*

(iv) Let $S_n = \sum_{k=1}^{n} \dfrac{\sin\left(\frac{k\pi}{2}\right)}{k + \sin\left(\frac{k\pi}{2}\right)}$.

(a) Show that, for $m \in \mathbb{Z}^+$, $S_{4m} = 0$. *[7 marks]*

(b) Show that $S_n \to 0$ as $n \to \infty$. *[2 marks]*

(c) Hence, or otherwise, show that the series converges as $n \to \infty$, and find its limit. *[1 mark]*

Euclidean Geometry and Conic Sections

10. *[Maximum mark: 30]*

(i) The focus of the parabola C is the point $\mathrm{F}(a, b)$ and the equation of the directrix is $x = -a$.

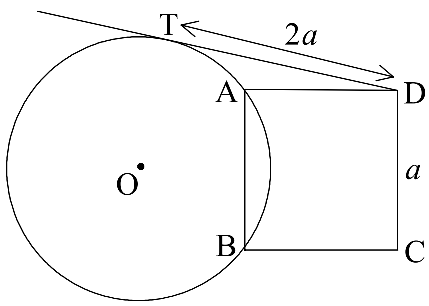
(a) (i) Find the equation of C from first principles.

(ii) Sketch C, marking the focus, directrix, axis of symmetry and vertex. *[6 marks]*

(b) The point P with x-coordinate $\frac{3a}{2}$ lies on the upper half of C. The tangent to C at P intersects the axis of symmetry of C at the point Q. The line through the vertex V of C perpendicular to the tangent (PQ) intersects (PQ) at the point R. Prove that $\mathrm{PR} : \mathrm{RQ} = 7 : 3$. *[12 marks]*

(c) The line through F parallel to (VR) intersects the line (PQ) at the point S. Find the coordinates of S. *[4 marks]*

(ii) The diagram shows a square ABCD of side a. A circle, centre O, radius r, passes through the vertices A and B. The length of the tangent to the circle from D is $2a$.



Find an expression for r in terms of a. *[8 marks]*